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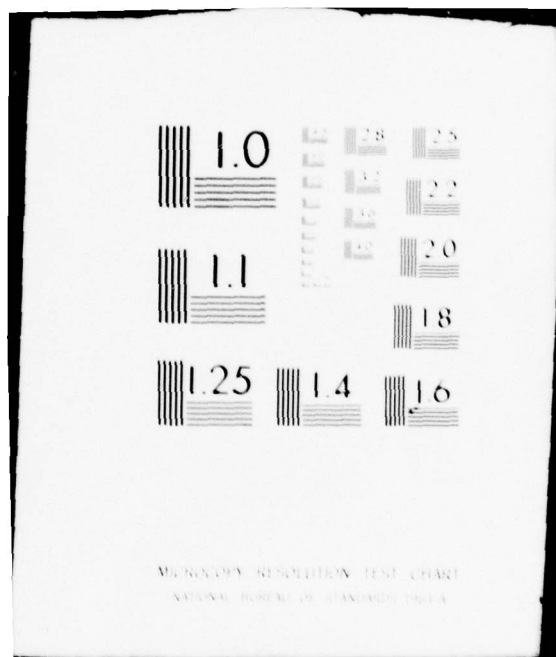
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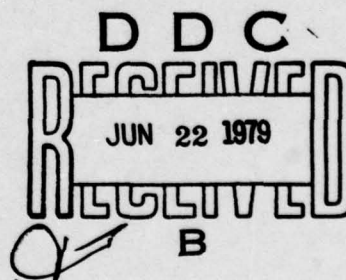
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In-House Report
April 1979



APPLICATION OF BAYESIAN TECHNIQUES TO RELIABILITY DEMONSTRATION, ESTIMATION AND UPDATING OF THE PRIOR DISTRIBUTION

Theodore S. Bolis

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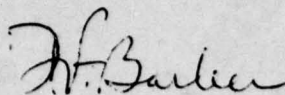
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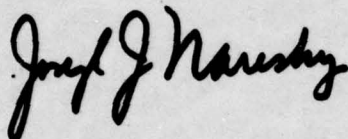
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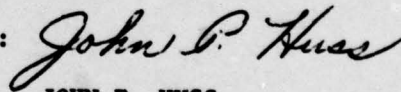
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A method is presented for the estimation of the shape and scale parameters of an inverted gamma prior distribution of the mean-time-to-failure for equipment having exponential time-to-failure distribution. This method, akin to the Maximum Likelihood Method, allows the use of all sorts of existing failure data on the equipment in question, provided a certain sufficient condition is satisfied. Further, this method (we call it the Generalized Maximum Likelihood Method) is usable to update the prior distribution, when new failure data become available. In the long run, this updating process will give rise to a solid			

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prior, which can confidently be used in Reliability Demonstration.

Various facets of the sufficient condition for the applicability of this estimation method are exposed, the variance-covariance matrix of the estimators is given under various randomness assumptions and some numerical considerations are discussed.

There is a brief discussion of alternate estimators in the case of a truncated test data.

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PREFACE

This report was prepared by Dr. Theodore S. Bolis, of the State University of New York at Oneonta, in residence at the Rome Air Development Center (RADC), under the 1978 USAF-ASEE Summer Faculty Research Program. The work presented is a part of an RADC program to develop Bayesian and other statistical techniques based on the use of prior data for practical application to Reliability Demonstration.

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1. INTRODUCTION

1.1 We consider equipment with exponentially distributed time-to-failure, i.e., the probability density function of the time-to-failure is given by

$$(1.1.1) \quad \phi(t|\theta) = \theta^{-1} \exp(-t/\theta), \quad t > 0; \theta > 0,$$

where the parameter θ is the mean-time-to-failure (MTTF) of the equipment.

We assume that θ itself has a prior distribution of the inverted gamma type, i.e., the prior probability density function of θ is given by

$$(1.1.2) \quad g(\theta; \lambda, \gamma) = \frac{\gamma^\lambda}{\Gamma(\lambda)} \theta^{-(\lambda+1)} \exp(-\gamma/\theta); \quad \theta > 0; \lambda > 0, \gamma > 0,$$

where λ is the shape parameter and γ is the scale parameter of the prior distribution.

1.2 Bayesian Reliability Test Plans based on the prior (1.1.2) have been developed by Schafer et al [4] and Goel [1] under various combinations of risks. The implementation of these plans require the estimated values of λ and γ in (1.1.2). Since the true MTTF θ of an equipment is not observable, we cannot directly fit existing data to the inverted gamma distribution (1.1.2). To get around this difficulty, we consider the probability function of the number of failures r in a fixed T , given θ . Because of the exponentiality assumption (1.1.1), this probability function is Poisson with parameter T/θ , i. e.

$$(1.2.1) \quad P_T(r|\theta) = \frac{1}{r!} (T/\theta)^r \exp(-T/\theta), \quad r = 0, 1, 2, \dots; T > 0.$$

Thus, the unconditional probability function of the number of failures r in a fixed time T is

$$(1.2.2) \quad P_T(r) = \int_0^\infty P_T(r|\theta) g(\theta; \lambda, \gamma) d\theta.$$

By using (1.1.2) and (1.2.1) and performing the integration, we obtain

$$(1.2.3) \quad P_T(r) = \binom{\lambda+r-1}{r} \left(\frac{\gamma}{T+\gamma}\right)^\lambda \left(\frac{T}{T+\gamma}\right)^r, \quad r = 0, 1, 2, \dots \text{ which}$$

is a negative binomial distribution with parameters λ and $T/(T+\gamma)$. If existing data on a type of equipment are of the form "number of failures in a fixed common time T ", then the parameters λ and γ can be estimated by using (1.2.3). Schafer et al [3] used the method of moments for this purpose, whereas Goel and Joglekar [2] used the maximum likelihood method.

1.3 An extreme and rather hypothetical case results when we keep the number of failures r fixed and observe the time T until the r th failure. Since T is the sum of r exponential random variables, its probability density is gamma with parameters r and θ^{-1} .

$$(1.3.1) \quad f_r(T|\theta) = \frac{\theta^{-r}}{(r-1)!} T^{r-1} \exp(-T/\theta), \quad T > 0$$

Thus, the unconditional probability density function of T is

$$(1.3.2) \quad f_r(T) = \int_0^\infty f_r(T|\theta) g(\theta; \lambda, \gamma) d\theta \\ = \frac{r}{T} \binom{\lambda+r-1}{r} \left(\frac{\gamma}{T+\gamma}\right)^\lambda \left(\frac{T}{T+\gamma}\right)^r, \quad T > 0.$$

This is just a scale transform of the inverted beta distribution written in this form to show its similarity with (1.2.3).

1.4 Existing failure data (especially field data) usually do not exhibit any of the two features discussed above. Usually the test or operational time varies from equipment to equipment of the same type. Thus, the data will usually be of the form (r_i, T_i) , $i=1, \dots, n$, where r_i is the number of failures of the i th equipment in time T_i . In a test situation, it is

feasible to control either r_i or T_i , but cost considerations recommend the control of T_i . Thus, it is desirable to estimate λ and γ in this more general situation, which encompasses the situations discussed in sections 1.2 and 1.3 as special cases. Schafer et al [3] present a method of estimation akin to the method of moments. This method however is not applicable if a single equipment had no failures at all.

1.5 In this report we present a general estimation method which we call The Generalized Maximum Likelihood Method. A sufficient condition for the existence of the estimators is given. In the case of fixed time data, it is shown that the condition is also necessary. The method has the advantage of being usable to update the prior when new data become available, e.g., from reliability demonstration tests.

If the data used for the estimation of the prior distribution are generated by a planned test, the estimability condition dictates ways of choosing (controlling) either the test times T_i or the number of failures r_i in such a way that the resulting Generalized Maximum Likelihood Equations have a solution, i.e., the estimators exist.

In the case of fixed time data, if the estimability condition is violated, some alternate estimation methods are presented.

2. THE GENERALIZED MAXIMUM LIKELIHOOD ESTIMATION METHOD

2.1 We suppose that n identical equipments with exponential time-to-failure distribution are tested in the following way: the i th equipment is tested for T_i hours, $i = 1, \dots, n$. Let r_i denote the number of failures of the i th equipment. We assume that the prior distribution of the MTTF θ is given by (1.1.2). Then, the unconditional probability function of r_i is given by (1.2.3), i.e.,

$$(2.1.1) \quad P_{T_i}(r_i) = \binom{\lambda + r_i - 1}{r_i} \left(\frac{\gamma}{T_i + \gamma} \right)^\lambda \left(\frac{T_i}{T_i + \gamma} \right)^{r_i}.$$

The Generalized Likelihood Function of the sample (r_i, T_i) , $i = 1, \dots, n$ is defined to be

$$(2.1.2) \quad L = \prod_{i=1}^n \binom{\lambda + r_i - 1}{r_i} \left(\frac{\gamma}{T_i + \gamma} \right)^\lambda \left(\frac{T_i}{T_i + \gamma} \right)^{r_i}$$

Just as in the classical Maximum Likelihood Estimation technique, the best explanation of the data (r_i, T_i) , $i = 1, \dots, n$ is provided by the values $(\hat{\lambda}, \hat{\gamma})$ of (λ, γ) at which the function L attains its maximum, if L has a maximum. As usual, in order to maximize L , it is enough to maximize its natural logarithm.

$$(2.1.3) \quad \ln L = \sum_{i=1}^n \ln \binom{\lambda + r_i - 1}{r_i} + \lambda \sum_{i=1}^n \ln \frac{\gamma}{T_i + \gamma} + \sum_{i=1}^n r_i \ln \frac{T_i}{T_i + \gamma}.$$

In order to obtain the critical point of L , we have to solve simultaneously the Generalized Likelihood Equations.

$$(2.1.4) \quad \frac{\partial}{\partial \lambda} \ln L = 0, \quad \frac{\partial}{\partial \gamma} \ln L = 0$$

which in our case become

$$(2.1.5) \quad \frac{\partial}{\partial \lambda} \ln L = \sum_{i=1}^n \left(\frac{1}{\lambda} + \dots + \frac{1}{\lambda + r_i - 1} \right) - \sum_{i=1}^n \ln \left(1 + \frac{T_i}{\gamma} \right) = 0$$

$$(2.1.6) \quad \frac{\partial}{\partial \gamma} \ln L = \frac{\lambda}{\gamma} \sum_{i=1}^n \frac{T_i}{T_i + \gamma} - \sum_{i=1}^n \frac{r_i}{T_i + \gamma} = 0$$

If we set $\alpha_j = \sum_{r_i \geq j} 1$ the above equations are reduced to

$$(2.1.7) \quad \sum_{j=1}^{\infty} \frac{\alpha_j}{\lambda + j - 1} = \sum_{i=1}^n \ln \left(1 + \frac{T_i}{\gamma} \right)$$

$$(2.1.8) \quad \lambda = \gamma \frac{\sum_{i=1}^n \frac{r_i}{T_i + \gamma}}{\sum_{i=1}^n \frac{T_i}{T_i + \gamma}}.$$

Since λ is given explicitly in terms of γ by (2.1.8), we can substitute in (2.1.7) to obtain an equation in γ alone. The resulting equation can be solved numerically (when a solution exists) to obtain the estimator $\hat{\gamma}$ and then, by means of (2.1.8) obtain the value $\hat{\lambda}$.

2.2 If we control the number of failures r_i and let T_i be random, the distribution of T_i is given by (1.3.2). It is immediate that the new Generalized Likelihood Function will be the same as the one given by (2.1.2) up to a factor

$$\prod_{i=1}^n r_i / T_i$$

which is independent of the parameters λ and γ . Therefore, the resulting

Generalized Likelihood Equations will be exactly the same as the ones given by (2.1.7) and (2.1.8). Thus, the Generalized Maximum Likelihood estimators will have the same form, irrespectively of whether we control the T_i 's or the r_i 's or any combination of them (e.g., controlling r_i for $i = 1, \dots, k$ and T_j for $j = k + 1, \dots, n$).

3. A SUFFICIENT CONDITION FOR THE EXISTENCE OF THE GENERALIZED MAXIMUM LIKELIHOOD ESTIMATORS

3.1 The system of equations (2.1.7) and (2.1.8) does not always have a solution. Although we could produce examples of actual data for which the Generalized Maximum Likelihood estimators do not exist, for simplicity's sake we resort to the following rather contrived

EXAMPLE 3.1.1 Let $n = 3$, $r_i = 0$, $r_2 = 1$, $r_3 = 2$, $T_1 = T_2 = T_3 = T$. Then the equations (2.1.7) and (2.1.8) are reduced to

$$(3.1.1) \quad \frac{2}{\lambda} + \frac{1}{\lambda+1} = 3 \ln \left(1 + \frac{T}{Y} \right)$$

$$(3.1.2) \quad \lambda = Y/T$$

whose simultaneous solution calls for the zero of the function

$$\Psi(\lambda) = \frac{2}{\lambda} + \frac{1}{\lambda+1} - 3 \ln \left(1 + \frac{1}{\lambda} \right), \quad \lambda > 0.$$

We claim that actually $\Psi(\lambda) > 0$ for all $\lambda > 0$. Indeed, $\lim_{\lambda \rightarrow 0+} \Psi(\lambda) = +\infty$ and $\lim_{\lambda \rightarrow \infty} \Psi(\lambda) = 0$

and thus, it is enough to show that Ψ is strictly decreasing. This is so since the derivative of Ψ is negative.

$$\Psi'(\lambda) = -\frac{2}{\lambda^2} - \frac{1}{(\lambda+1)^2} + \frac{3}{\lambda(\lambda+1)} = -\frac{\lambda+2}{\lambda^2(\lambda+1)^2} < 0$$

3.2 We now give a sufficient condition for the solvability of the Generalized Likelihood Equations (2.1.7) and (2.1.8). We use the following notation:

$$\bar{r} = \frac{1}{n} \sum_{i=1}^n r_i, \quad S_r^2 = \frac{1}{n} \sum_{i=1}^n r_i^2 - \bar{r}^2,$$

$$\bar{T} = \frac{1}{n} \sum_{i=1}^n T_i, \quad S_T^2 = \frac{1}{n} \sum_{i=1}^n T_i^2 - \bar{T}^2,$$

$$\text{Cov}(r, T) = \frac{1}{n} \sum_{i=1}^n r_i T_i - \bar{r} \bar{T}.$$

We shall prove the following

THEOREM 3.2.1 If

$$(C) \quad 2 \bar{r} \bar{T} \text{cov}(r, T) < \bar{T}^2 (S_r^2 - \bar{r}) + \bar{r}^2 S_T^2,$$

then the Generalized Likelihood Equations (2.1.7) and (2.1.8) have a solution.

PROOF. It suffices to show that the function defined by

$$(3.2.1) \quad W(\gamma) = \sum_{j \geq 1} \frac{\alpha_j}{\lambda(\gamma) + j - 1} - \sum_{i=1}^n \ln \left(1 + \frac{T_i}{\gamma} \right), \quad \gamma > 0$$

where

$$\lambda(\gamma) = \gamma \frac{\sum_{i=1}^n \frac{r_i}{T_i + \gamma}}{\sum_{i=1}^n \frac{T_i}{T_i + \gamma}}$$

has a zero. We observe that $\alpha_j \neq 0$, because, otherwise, all $\alpha_j = 0$ which implies that all $r_i = 0$ and the condition (C) violated (it is reduced to $0 < 0$). Since $\lambda(\gamma)$ tends to zero when γ tends to zero and since $\alpha_1 > 0$ we get $\lim_{\gamma \rightarrow 0+} W(\gamma) = +\infty$.

Thus, it suffices to show that $W(\gamma)$ is negative for large γ . To this end we observe that

$$(3.2.2) \quad \lambda(\gamma) = \gamma \left[\sum_{i=1}^n r_i - \sum_{i=1}^n r_i T_i \gamma^{-1} + o(\gamma^{-1}) \right] / \left[\sum_{i=1}^n T_i - \sum_{i=1}^n T_i^2 \gamma^{-1} + o(\gamma^{-1}) \right]$$

$$= \frac{\bar{r}}{\bar{T}} \gamma + \frac{\bar{r} S_T^2 - \bar{T} \text{cov}(r, T)}{\bar{T}^2} + o(1) \text{ as } \gamma \rightarrow +\infty$$

Substituting (3.2.2) into (3.2.1) we obtain

(3.3.3)

$$W(\gamma) = \sum_{j \geq 1} \alpha_j \left/ \left[-\frac{\bar{r}}{\bar{T}} \gamma + \frac{\bar{r} S_T^2 - \bar{T} \text{cov}(r, T)}{\bar{T}^2} + j - 1 + o(1) \right] \right/ - \sum_{i=1}^n \ln \left(1 + \frac{T_i}{\gamma} \right)$$

$$= \frac{\bar{T}}{\bar{r}} \gamma^{-1} \left[\sum_{j \geq 1} \alpha_j - \left[\frac{\bar{r} S_T^2 - \bar{T} \text{cov}(r, T)}{\bar{r} \bar{T}} \sum_{j \geq 1} \alpha_j + \frac{\bar{T}}{\bar{r}} \sum_{j \geq 1} (j-1) \alpha_j \right] \gamma^{-1} + \right.$$

$$\left. + o(\gamma^{-1}) \right] - \sum_{i=1}^n T_i \gamma^{-1} + \frac{1}{2} \sum_{i=1}^n T_i^2 \gamma^{-2} + o(\gamma^{-2})$$

as $\gamma \rightarrow +\infty$

Since $\sum_{j \geq 1} \alpha_j = n\bar{r}$ and $\sum_{j \geq 1} (j-1) \alpha_j = \frac{1}{2} (S_r^2 + \bar{r}^2 - \bar{r})$,

(3.3.3) is reduced to

$$W(\gamma) = -\frac{n}{2\bar{r}} \left(-2\bar{r} \bar{T} \text{cov}(r, T) + \bar{T}^2 (S_r^2 - \bar{r}) + \bar{r}^2 S_T^2 \right) \gamma^{-2} + o(\gamma^{-2})$$

as $\gamma \rightarrow +\infty$

Because of the condition (C), $W(\gamma) < 0$ for large λ and the theorem is proved.

3.3 We observe that in case $T_i = T$ for all $i=1, \dots, n$, the condition (C)

is reduced to (C₁) $\bar{r} < S_r^2$

which is exactly the condition for the applicability of the method of moments to the estimation of the parameters of a negative binomial distribution. In the next section we will show that the condition (C_1) is also necessary for Maximum Likelihood estimability in the case of the negative binomial distribution.

If, on the other hand, $r_i = r$ for all $i = 1, \dots, n$, then the condition (C) is reduced to

$$(C_2) \quad \bar{T} < \sqrt{r} S_T$$

which is exactly the condition for the applicability of the method of moments to the estimation of the parameters of the inverted beta distribution (1.3.2). Of course, if the method of moments is applied to this situation, the value of $\hat{\lambda}$ will always be greater than 2, because this distribution does not have a second moment if $\lambda \leq 2$.

Another remark about the condition (C) is in order. Since the values of T_i are usually large, both sides of the inequality become rather large. It is, therefore, convenient to write condition (C) in the following equivalent form (3.3.1)

$$\bar{r} < \frac{1}{n} \sum_{i=1}^n \left(r_i - \frac{\bar{r}}{\bar{T}} T_i \right)^2$$

3.4 We now prove the following

THEOREM 3.4.1 The Maximum Likelihood estimators of the parameters of the negative binomial distribution (1.2.3) exist if and only if $\bar{r} < S_r^2$.

The sufficient part of this theorem is contained in the Theorem 3.2.1.

For the necessity part we need the following lemma:

LEMMA 3.4.2 For all positive integers n and all positive λ we have

$$(3.4.1) \quad \sum_{j=1}^n \frac{1}{(\lambda+j-1)^2} \geq \frac{n}{\lambda(\lambda+n-1)}.$$

The inequality is strict if $n > 1$.

The proof of the lemma is inductive. The inequality is obviously true for $n = 1$. Assuming it true for n , we shall prove it for $n + 1$. Indeed

$$\sum_{j=1}^{n+1} \frac{1}{(\lambda+j-1)^2} = \sum_{j=1}^n \frac{1}{(\lambda+j-1)^2} + \frac{1}{(\lambda+n)^2} \geq \frac{n}{\lambda(\lambda+n-1)} +$$

$$\frac{1}{(\lambda+n)^2} = \frac{n+1}{\lambda(\lambda+n)} + \frac{n}{\lambda(\lambda+n-1)(\lambda+n)^2} > \frac{n+1}{\lambda(\lambda+n)}.$$

This completes the proof of the lemma.

PROOF OF THE THEOREM: We need only prove the necessity of the condition.

Since $T_i = T$ for all $i = 1, \dots, n$, the equations (2.1.5) and (2.1.6) are reduced to

$$(3.4.2) \quad \sum_{i=1}^n \left(\frac{1}{\lambda} + \dots + \frac{1}{\lambda+r_i-1} \right) - n \ln \left(1 + \frac{T}{Y} \right) = 0$$

$$(3.4.3) \quad T/Y = \bar{r}/\lambda.$$

Obviously, it is enough to prove that the function Ψ defined by

$$\Psi(\lambda) = \sum_{i=1}^n \left(\frac{1}{\lambda} + \dots + \frac{1}{\lambda+r_i-1} \right) - n \ln \left(1 + \frac{\bar{r}}{\lambda} \right), \quad \lambda > 0$$

has no zeros if $\bar{r} \geq S_r^2$. If all r_i are zero, then the function Ψ

is identically zero, the log-likelihood function (2.1.3) is reduced to

$$n\lambda \ln \frac{\gamma}{T+\gamma}$$

and it is obvious that this function has no maximum in the range of the parameters. Thus, we may assume that at least one r_i is different than zero. Then

$$\lim_{\lambda \rightarrow 0+} \Psi(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} \Psi(\lambda) = 0.$$

Hence, in order to show that the function Ψ has no zeros, it suffices to show that its derivative is nonpositive. By using the inequality (3.4.1) we get

$$\begin{aligned} (3.4.4) \quad \Psi'(\lambda) &= - \sum_{i=1}^n \left(\frac{1}{\lambda^2} + \dots + \frac{1}{(\lambda+r_i-1)^2} \right) + \frac{n\bar{r}}{\lambda(\lambda+\bar{r})} \\ &\leq - \sum_{i=1}^n \frac{r_i}{\lambda(\lambda+r_i-1)} + \frac{n\bar{r}}{\lambda(\lambda+\bar{r})}. \end{aligned}$$

We now use the convexity of the function

$$w(x) = 1/(\lambda-1+x), \quad x \geq 1; \quad \lambda > 0$$

with the r_i 's in the numerator of the summand of the right hand side of the inequality (3.4.4) used as weights and the r_i 's in the denominator as points in the domain of w . The so-called Jensen's inequality yields

$$\sum_{i=1}^n \frac{r_i}{\lambda+r_i-1} \geq \frac{\sum r_i}{\lambda-1 + \sum r_i^2 / \sum r_i} = \frac{n\bar{r}^2}{\lambda\bar{r} + S_r^2 + \bar{r}^2 - \bar{r}}.$$

Therefore, going back to (3.4.4) we get

$$\Psi'(\lambda) \leq - \frac{1}{\lambda} \frac{n\bar{r}^2}{\lambda\bar{r} + S_r^2 + \bar{r}^2 - \bar{r}} + \frac{n\bar{r}}{\lambda(\lambda+\bar{r})} =$$

$$= - \frac{n \bar{r} (\bar{r} - s_r^2)}{\lambda(\lambda + \bar{r}) (\lambda \bar{r} + s_r^2 + \bar{r}^2 - \bar{r})} \leq 0, \lambda > 0$$

since

$$\bar{r} \geq s_r^2, s_r^2 + \bar{r}^2 - \bar{r} = \frac{2}{n} \sum_{j \geq 1} (j-1) \alpha_j \geq 0.$$

This completes the proof of the theorem.

4. CONTROLLING FOR ESTIMABILITY

4.1 If we let $n = 2m$, $T_i = T$ for $i = 1, \dots, m$ and $T_i = kT$ for $i = m + 1, \dots, 2m$, $k > 1$, then

$$(4.1.1) \quad \bar{T} = \frac{k+1}{2} T, \quad S_T^2 = \left(\frac{k-1}{2}\right)^2 T^2, \quad \text{cov}(r, T) = \frac{k-1}{2} T \left[\frac{1}{m} \sum_{i=m}^{2m} r_i - \bar{r} \right]$$

By substituting (4.1.1) into (C) we obtain

$$(4.1.2) \quad \frac{2}{m} \bar{r} \sum_{i=m}^{2m} r_i < \frac{k+1}{k-1} (S_r^2 - \bar{r}) + \frac{3k+1}{k+1} \bar{r}^2,$$

a condition independent of T . This kind of test designing enhances the possibility of having $S_r^2 > \bar{r}$ and the condition (C) satisfied. In particular, if $k = 2$, the condition (4.1.2) is reduced to

$$(4.1.3) \quad \frac{2}{m} \bar{r} \sum_{i=m}^{2m} r_i < 3 (S_r^2 - \bar{r}) + \frac{7}{2} \bar{r}^2.$$

4.2 In the more expensive case of controlling the number of failures and letting the test time be random, we can always assure $\bar{r} < S_r^2$. We consider the following design: $n = 2m$, $r_i = r$ for $i = 1, \dots, m$ and $r_i = kr$ for $i = m + 1, \dots, 2m$, $k > 1$. Then

$$(4.2.1) \quad \bar{r} = \frac{k+1}{2} r, \quad S_r^2 = \left(\frac{k-1}{2}\right)^2 r^2, \quad \text{cov}(r, T) = \frac{k-1}{2} r \left[\frac{1}{m} \sum_{i=m}^{2m} T_i - \bar{T} \right]$$

and the condition (C) is reduced to

$$(4.2.2) \quad \frac{2}{m} \bar{T} \sum_{i=m}^{2m} T_i < \bar{T}^2 \left[\frac{k-1}{k+1} r + 2 \frac{k-2}{k-1} \right] + \frac{k-1}{k+1} S_T^2.$$

By choosing $k = 2$ and $r = 12$, or $k = 3$ and $r = 7$, or $k = 4$ and $r = 5$, or $k = 5$ and $r = 4$, this condition is always satisfied. Of course, one does not have to go to such extremes. For $k = 2$ and $r = 6$ for example, the condition will usually be satisfied.

5. VARIABILITY OF THE ESTIMATORS

5.1 The variance-covariance matrix of the Generalized Maximum Likelihood estimators $\hat{\lambda}$ and $\hat{\gamma}$ is given by

$$(5.1.1) \quad \begin{bmatrix} \text{var}(\hat{\lambda}) & \text{cov}(\hat{\lambda}, \hat{\gamma}) \\ \text{cov}(\hat{\lambda}, \hat{\gamma}) & \text{var}(\hat{\gamma}) \end{bmatrix} = - \begin{bmatrix} E \frac{\partial^2 \ln L}{\partial \lambda^2} & E \frac{\partial^2 \ln L}{\partial \lambda \partial \gamma} \\ E \frac{\partial^2 \ln L}{\partial \lambda \partial \gamma} & E \frac{\partial^2 \ln L}{\partial \gamma^2} \end{bmatrix}^{-1}$$

From (2.1.5) and (2.1.6) we obtain

$$(5.1.2) \quad \frac{\partial^2}{\partial \lambda^2} \ln L = - \sum_{i=1}^n \left(\frac{1}{\lambda^2} + \dots + \frac{1}{(\lambda + r_i - 1)^2} \right) = - \sum_{j=1}^{\alpha_j} \frac{1}{(\lambda + j - 1)^2}$$

$$(5.1.3) \quad \frac{\partial^2}{\partial \lambda \partial \gamma} \ln L = \frac{1}{\gamma} \sum_{i=1}^n \frac{T_i}{T_i + \gamma}$$

$$(5.1.4) \quad \frac{\partial^2}{\partial \gamma^2} \ln L = - \frac{\lambda}{\gamma^2} \sum_{i=1}^n \frac{T_i (2\gamma + T_i)}{(T_i + \gamma)^2} + \sum_{i=1}^n \frac{r_i}{(T_i + \gamma)^2}$$

5.2 Assuming the T_i non-random and the r_i random, we obtain

$$(5.2.1) \quad E \frac{\partial^2}{\partial \lambda^2} \ln L = - \sum_{i=1}^n F \left(\frac{T_i}{T_i + \gamma}; \lambda \right)$$

where

$$(5.2.2) \quad F(p; \lambda) = \sum_{j=1}^{\infty} p^j / j^2 \binom{\lambda+j-1}{j};$$

$$(5.2.3) \quad E \frac{\partial^2}{\partial \lambda \partial \gamma} \ln L = \frac{1}{\gamma} \sum_{i=1}^n \frac{T_i}{T_i + \gamma};$$

$$(5.2.4) \quad E \frac{\partial^2}{\partial \gamma^2} \ln L = - \frac{\lambda}{\gamma^2} \sum_{i=1}^n \frac{T_i}{T_i + \gamma}$$

By substituting (5.2.1), (5.2.3) and (5.2.4) in (5.1.1) we get

$$(5.2.5) \quad \text{var}(\hat{\lambda}) = \lambda / \left[\lambda \sum_{i=1}^n F\left(\frac{T_i}{T_i + \gamma}; \lambda\right) - \sum_{i=1}^n \frac{T_i}{T_i + \gamma} \right];$$

$$(5.2.6) \quad \text{var}(\hat{\gamma}) = \gamma^2 \sum_{i=1}^n F\left(\frac{T_i}{T_i + \gamma}; \lambda\right) / \sum_{i=1}^n \frac{T_i}{T_i + \gamma} \left[\lambda \sum_{i=1}^n F\left(\frac{T_i}{T_i + \gamma}; \lambda\right) - \sum_{i=1}^n \frac{T_i}{T_i + \gamma} \right];$$

$$(5.2.7) \quad \text{cov}(\hat{\lambda}, \hat{\gamma}) = \gamma / \left[\lambda \sum_{i=1}^n F\left(\frac{T_i}{T_i + \gamma}; \lambda\right) - \sum_{i=1}^n \frac{T_i}{T_i + \gamma} \right].$$

5.3 By assuming T_i random and r_i non-random, we obtain

$$(5.3.1) \quad E \frac{\partial^2}{\partial \lambda^2} \ln L = - \sum_{j \geq 1} \frac{\alpha_j}{(\lambda + j - 1)^2};$$

$$(5.3.2) \quad E \frac{\partial^2}{\partial \lambda \partial \gamma} \ln L = \frac{1}{\gamma} \sum_{i=1}^n \frac{r_i}{\lambda + r_i};$$

$$(5.3.3) \quad E \frac{\partial^2}{\partial \gamma^2} \ln L = - \frac{\lambda}{\gamma^2} \sum_{i=1}^n \frac{r_i}{\lambda + r_i + 1}.$$

Substituting into (5.1.1), we get

$$(5.3.4) \quad \text{var}(\hat{\lambda}) = \lambda \sum_{i=1}^n \frac{r_i}{\lambda + r_i + 1} / \Delta;$$

$$(5.3.5) \quad \text{var}(\hat{\gamma}) = \gamma^2 \sum_{j=1}^n \frac{\alpha_j}{(\lambda + j - 1)^2} / \Delta;$$

$$(5.3.6) \quad \text{cov}(\hat{\lambda}, \hat{\gamma}) = \gamma \sum_{i=1}^n \frac{r_i}{\lambda + r_i} / \Delta,$$

where

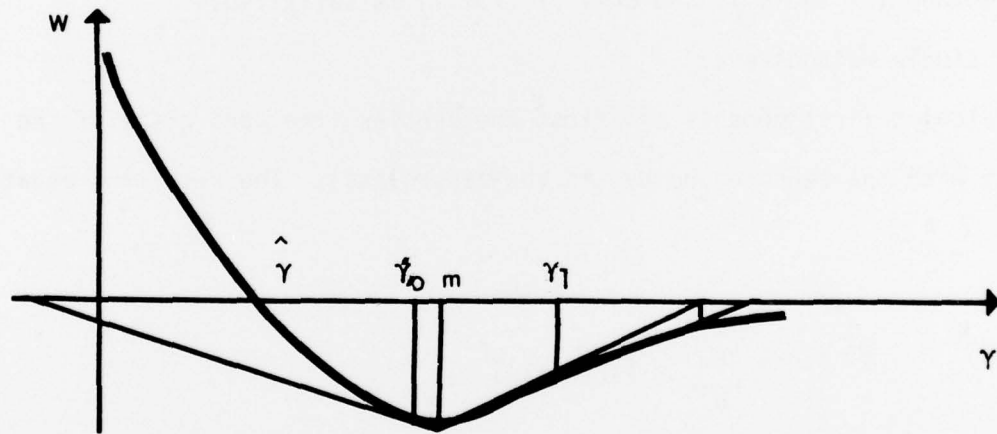
$$(5.3.7) \quad \Delta = \lambda \sum_{j=1}^n \frac{\alpha_j}{(\lambda + j - 1)^2} \sum_{i=1}^n \frac{r_i}{\lambda + r_i + 1} - \left(\sum_{i=1}^n \frac{r_i}{\lambda + r_i} \right)^2.$$

The case of mixed controls can be handled similarly. In order to estimate these variances and covariances, we substitute in the above formulas the estimated values $\hat{\lambda}$ and $\hat{\gamma}$ instead of λ and γ .

6. NUMERICAL CONSIDERATIONS

6.1 In writing up a computer program for the numerical solution of the Generalized Maximum Likelihood Equations (2.1.7) and (2.1.8) the following observations should be taken into account. The solution of these equations is reduced to finding the zero of the function W defined by (3.2.1). The shape of this function is given by figure 1.

If the Newton-Raphson iterative method is employed, care should be exercised in choosing the initial value. If the initial value is greater than the minimum m of the function W , the Newton-Raphson process will diverge to infinity (initial value γ_1 in Figure 1), whereas, if the initial value is less than m but near m , the first iteration will produce a negative value for γ (initial value γ_0 in Figure 1). Therefore, an initial value, at which W is positive, should be chosen, if the Newton-Raphson method is to be used. Because of the complexity of the derivation of W and since only nearest integer accuracy is required for γ , some slower converging interpolative method may be more suitable.



7. ALTERNATE ESTIMATORS (NEGATIVE BINOMIAL)

7.1 In the case of fixed time testing (i.e. $T_i = T$ for all $i = 1, \dots, n$), we saw that the distribution of the number of failures is the negative binomial given by (1.2.3) and that the Maximum Likelihood estimators exist if and only if

$$(7.1.1) \quad \bar{r} < S_r^2$$

by the theorem 3.4.1. Since the solution of the Likelihood Equations require numerical techniques, several alternate methods are often used. Among them, the method of moments is the most popular. It yields

$$(7.1.2) \quad \hat{\lambda} = \bar{r}^2 / (S_r^2 - \bar{r}), \quad \hat{\gamma} = \bar{r} T / (S_r^2 - \bar{r})$$

and it is highly efficient in a wide range of the parameters. Of course, this method is usable if and only if (7.1.1) is satisfied.

Other simple methods are:

(A) Matching first moments and first frequencies (the zero class of the sample with the expected number in the zero class). The resulting equations are

$$(7.1.3) \quad \lambda T = \bar{r} \gamma, \quad \frac{n_0}{n} = (\lambda / (r + \lambda))^\lambda,$$

where n_0 is the zero class of the sample. It is not hard to prove that this method is usable if and only if

$$(7.1.4) \quad \bar{r} > \ln(n/n_0).$$

(B) Matching first moments and the ratio of the first two frequencies.

The resulting estimators are

$$(7.1.5) \quad \hat{\lambda} = n_0 \bar{r} / (n_1 \bar{r} - n_0), \quad \hat{\gamma} = n_0 T / (n_1 \bar{r} - n_0).$$

Obviously this method is usable if and only if

$$(7.1.6) \quad \bar{r} > n_0/n_1.$$

The efficiency of these two methods has been investigated by Katti and Gurland [5]. They found that there are ranges of the parameters, where these methods are superior to the method of moments.

7.2 Another estimation investigated by Katti and Gurland [5] is the minimum chi-squared method. It is a highly efficient and rather complicated method for which numerical techniques are required. We did not attempt to find necessary or sufficient conditions for the applicability of this method.

8. CONCLUSIONS AND RECOMMENDATIONS

In this report we presented a method for estimating the shape and scale parameters of an inverted gamma prior distribution of the mean-time-to-failure for equipment having exponential time-to-failure distribution. All sorts of existing failure data on the equipment in question are usable provided a certain sufficient condition is satisfied. Further, the method can be used to update the prior when new failure data become available. This periodic updating will give rise to a solid prior which can confidently be used in Reliability Demonstration.

It is recommended that a computer program be written to solve the Generalized Likelihood Equations that define these estimators and to compute

their variance-covariance matrix. To this end, the recommendations put forward in Section 6 should be taken into account. The program can then be used for the periodic updating of the prior distribution.

Examples of the application of the theory developed in this report are presented in the Appendix.

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APPENDIX

In this appendix we present some examples of failure data, for which the theory developed in this report is applied.

EXAMPLE 1. Three pieces of the equipment AN/ASK-6 were tested with the following results (time measured in hours):

$$r_1 = 4, T_1 = 1961; r_2 = 2, T_2 = 1814; r_3 = 1, T_3 = 1890$$

thus $\bar{r} = 7/3$, $\bar{T} = 5665/3$ and the right hand side of (3.3.1) is 1.44271, whereas the left hand side is 2.3333. Therefore, the condition (C) is not satisfied.

EXAMPLE 2. Equipment: AN/ASN-108, $n=3$, $r_1 = 3$, $T_1 = 1522$; $r_2 = 0$, $T_2 = 1725$; $r_3 = 1$, $T_3 = 997$

It is easily verified that the condition (3.3.1) is satisfied ($1.333 < 1.699$). Obviously $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = 1$ and $\alpha_j = 0$ for $j \geq 4$, and the generalized maximum likelihood equations (2.1.7) and (2.1.8) become:

$$(A1) \quad W(\gamma) = \frac{2}{\lambda(\gamma)} + \frac{1}{\lambda(\gamma)+1} + \frac{1}{\lambda(\gamma)+2} - \ln\left(1 + \frac{1522}{\gamma}\right) - \ln\left(1 + \frac{1725}{\gamma}\right) - \ln\left(1 + \frac{997}{\gamma}\right) = 0$$

where

$$(A2) \quad \lambda(\gamma) = \gamma \left(\frac{3}{1522 + \gamma} + \frac{1}{997 + \gamma} \right) / \left(\frac{1522}{1522 + \gamma} + \frac{1725}{1725 + \gamma} + \frac{997}{997 + \gamma} \right)$$

In order to find the estimators $\hat{\lambda}$ and $\hat{\gamma}$, we start with a moderate initial value $\gamma_0 = 1000$ and find $W(1000) = .267688$. Since $W(\gamma_0) > 0$ we try $\gamma_1 = 2000$ and find $W(2000) = .039124$. By linear extrapolation we

try the next value:

$$\gamma_2 = \gamma_1 \frac{\gamma_1 - \gamma_0}{W(\gamma_1) - W(\gamma_0)} \quad W(\gamma_1) = 2,171.17$$

and find $W(\gamma_2) = .028774$. Linear extrapolation produces

$$\gamma_3 = \gamma_2 - \frac{\gamma_2 - \gamma_1}{W(\gamma_2) - W(\gamma_1)} W(\gamma_2) = 2,647.04$$

with $W(\gamma_3) = .011590$. Continuing this way, we find $\gamma_4 = 2968$

$W(\gamma_4) = .005417$, $\gamma_5 = 3250$, $W(\gamma_5) = .001931$, $\gamma_6 = 3406$, $W(\gamma_6) = .000525$

$\gamma_7 = 3464$, $W(\gamma_7) = 76.1807 \cdot 10^{-6}$, $\gamma_8 = 3474$, $W(\gamma_8) = 24.177 \cdot 10^{-7}$

$W(3475) < 0$. Actually $\hat{\gamma} = 3474$ $\hat{\lambda} = 3.332$.

In this example, we approximated γ by staying on the left side of its true value and by using linear extrapolation. Of course we could have started with, say $\gamma_0 = 2000$, $\gamma_1 = 4000$, observe that $W(\gamma_0) > 0$ and $W(\gamma_1) < 0$ and use linear interpolation to find γ_3 and continue this way. Of course in this example, the number of the equipments is too small to rely on the estimators. It is only presented for the purpose of demonstrating the numerical technique. The variability of $\hat{\lambda}$ and $\hat{\gamma}$ can be computed from the formulas (5.2.5), (5.2.6) and (5.2.7) if we assume that the T_i 's are controlled and the r_i 's are random. This assumption was not exactly met in this experiment, but it is more realistic than the other way around. We find

standard deviation of $\hat{\lambda} = 10.27$

standard deviation of $\hat{\gamma} = 10,903.56$

covariance of $(\hat{\lambda}, \hat{\gamma}) = 109,985.92$

correlation coefficient of $(\hat{\lambda}, \hat{\gamma}) = .9821$

Thus, the variability of the estimators is extremely high, which is expected in view of the low value of the number of equipments tested.

EXAMPLE 3. The following field failure data of a Ground Electronic Digital Processing System were collected:

	r	T		r	T
1.	3	5068.6	17.	6	6435.2
2.	15	7486.2	18.	7	4624.5
3.	10	7587.4	19.	14	5327.0
4.	5	4978.4	20.	11	7486.2
5.	7	6000.0	21.	3	6271.7
6.	5	5187.5	22.	9	6934.5
7.	3	7808.5	23.	16	7114.8
8.	3	2246.4	24.	9	7626.1
9.	7	4735.2	25.	7	4372.2
10.	13	7670.9	26.	10	5409.2
11.	3	4320.2	27.	8	5617.6
12.	3	3599.1	28.	4	2844.8
13.	6	7865.8	29.	10	1976.3
14.	1	1941.5	30.	7	1987.3
15.	21	7273.6	31.	3	4952.5
16.	17	6891.8			

The condition (3.3.1) is satisfied. The values of the α 's are $\alpha_1 = 31$, $\alpha_2 = 30$, $\alpha_3 = 30$, $\alpha_4 = 23$, $\alpha_5 = 22$, $\alpha_6 = 20$, $\alpha_7 = 18$, $\alpha_8 = 13$, $\alpha_9 = 12$, $\alpha_{10} = 10$, $\alpha_{11} = 7$, $\alpha_{12} = 6$, $\alpha_{13} = 6$, $\alpha_{14} = 5$, $\alpha_{15} = 4$, $\alpha_{16} = 3$, $\alpha_{17} = 2$, $\alpha_{18} = \alpha_{19} = \alpha_{20} = \alpha_{21} = 1$.

By starting with the values $\gamma_0 = 3000$ and $\gamma_1 = 5000$, we obtain

$$\begin{array}{lll} \gamma_0 = 3000 & \lambda_0 = 4.4313261 & w_0 = .41345261 \\ \gamma_1 = 5000 & \lambda_1 = 7.3422691 & w_1 = -.07330695 \end{array}$$

$$\gamma_2 = \gamma_1 - w_1 (\gamma_1 - \gamma_0) / (w_1 - w_0) \approx 4700$$

$$\lambda_2 = 6.9017913 \quad w_2 = -.0328700$$

$$\gamma_3 = \gamma_2 - w_2 (\gamma_2 - \gamma_1) / (w_2 - w_1) \approx 4456$$

$$\lambda_3 = 6.5514034 \quad w_3 = -.0128367$$

$$\gamma_4 = \gamma_3 - w_3 (\gamma_3 - \gamma_2) / (w_3 - w_2) \approx 4300$$

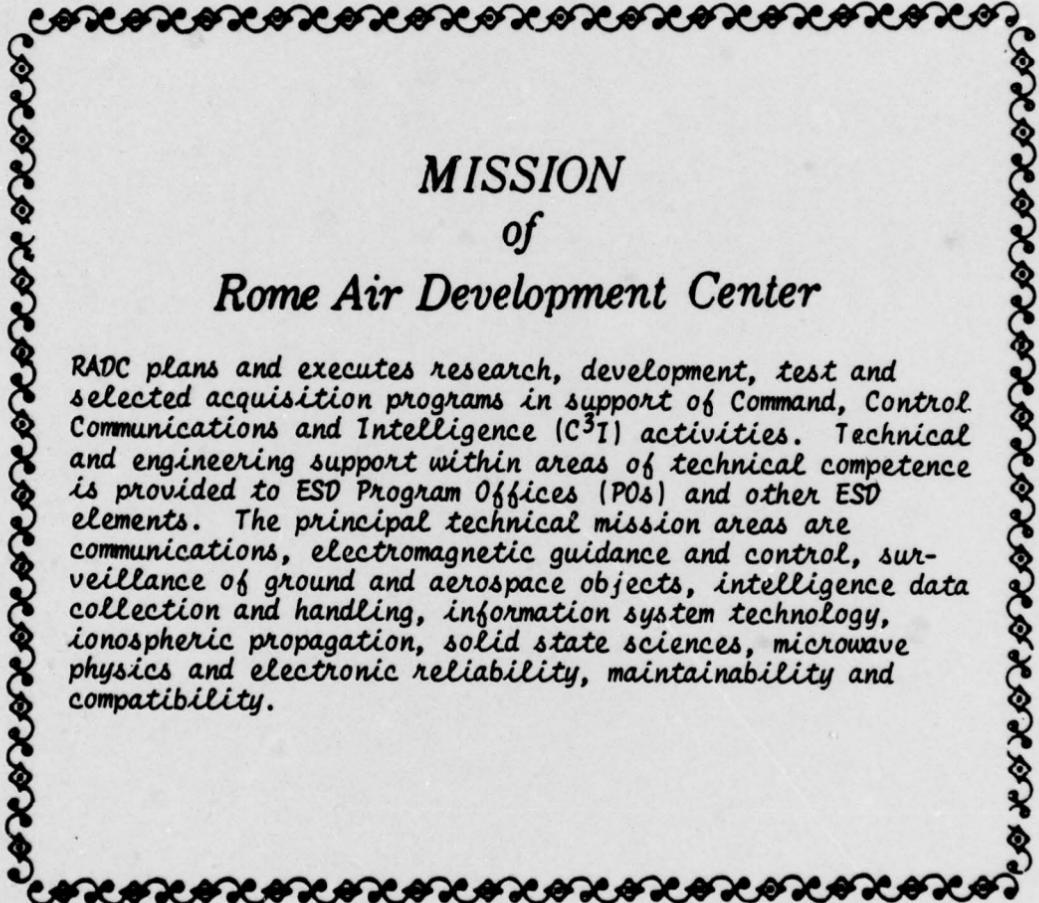
$$\lambda_4 = 6.3297434 \quad w_4 = -.0040735$$

Since calculations on a pocket calculator are very tedious, we stop by just extrapolating once more. Thus,

$$\hat{\gamma} = \gamma_4 - w_4 (\gamma_4 - \gamma_3) / (w_4 - w_3) \approx 4222$$

$$\hat{\lambda} = \lambda_4 - w_4 (\lambda_4 - \lambda_3) / (w_4 - w_3) \approx 6.227$$

The variability of these estimators is relatively low. For example, the s.d. of $\hat{\lambda}$ computed by means of the formula (5.2.5) is approximately 2.918. Large test times and large number of equipments tested naturally decrease the variance of the estimators.



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